

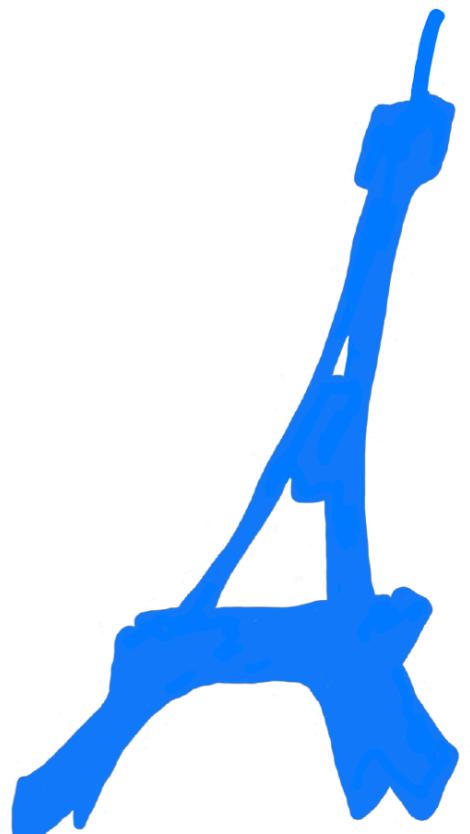
# Further exploitation of the RB framework

Yvon Maday,

Laboratoire Jacques-Louis Lions  
Sorbonne Université, Paris, Roscoff, France,  
Institut Universitaire de France

Providence — February 2020

Mathematics of Reduced Order Models



**SORBONNE  
UNIVERSITÉ**  
CRÉATEURS DE FUTURS  
DEPUIS 1257



**institut  
universitaire  
de France**

Reduced Basis Methods is one of the way  
for Model Reduction

# Vague Statements

The idea is to use the fact that  
the « state » we are interested in  
is described by a quantity  $u(x, t; \mu)$   
that is a function  
depending on space (and time) and a parameter  $\mu$

# Parametric model manifold

we introduce the set of all solutions to the mathematical model

$$\mathcal{S} = \{u(., \mu), \mu \in \mathcal{D}\}$$

and assume it has a small Kolmogorov n-width

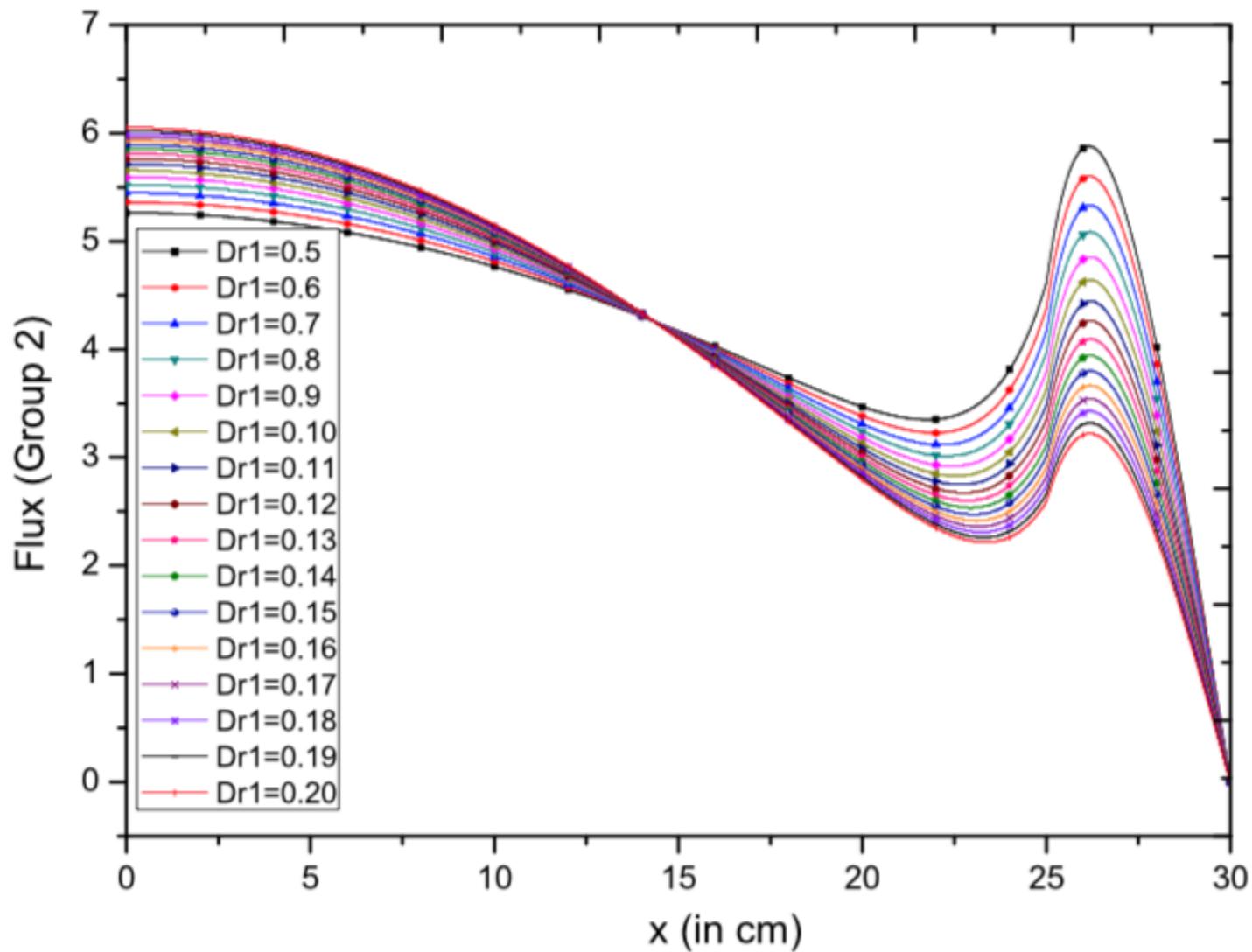
## **small Kolmogorov n-width**

means that there exists a small set of functions in  $\mathcal{S}$   
or in  $\text{Span} \{ \mathcal{S} \}$

such that, any  $u$  in  $\mathcal{S}$

is well approximated by a linear combination of these few functions

an example



# Kolmogorov n-width

**Definition** Let  $X$  be a normed linear space,  $\mathcal{S}$  be a subset of  $X$  and  $X_n$  be a generic  $n$ -dimensional subspace of  $\mathcal{X}$ . The deviation of  $\mathcal{S}$  from  $X_n$  is

$$E(\mathcal{S}; X_n) = \sup_{u \in \mathcal{S}} \inf_{v_n \in X_n} \|u - v_n\|_X.$$

The *Kolmogorov n-width* of  $\mathcal{S}$  in  $X$  is given by

$$d_n(\mathcal{S}, X) = \inf_{X_n} \sup_{u \in \mathcal{S}} \inf_{v_n \in X_n} \|u - v_n\|_X$$

The  $n$ -width of  $\mathcal{S}$  thus measures the extent to which  $\mathcal{S}$  may be approximated by a  $n$ -dimensional subspace of  $X$ .

How to get the Kolmogorov best space  $X_n$  ??

# How to get the Kolmogorov best space $X_n$ ??

$X_n$  optimal space is not attainable :

an approximation can be given by PCA/SVD  
... based on some orthogonal decomposition

another way is through greedy approach

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

a) possibly measures, either pointwize  $u(x_i, t_k, \mu)$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either pointwize  $u(x_i, t_k, \mu)$   
or moments  $\int \varphi_{i,k}(x, t)u(x, t, \mu)dxdt$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either pointwize  $u(x_i, t_k, \mu)$   
or moments  $\int \varphi_{i,k}(x, t)u(x, t, \mu)dxdt$   
or
- b) possibly a mathematical model for the behaviour of the phenomenon, depending on the parameter  $\mu$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either
- Reconstruction by interpolation  
moments  $\int \varphi_{i,k}(x, t) u(x, t, \mu) dx dt$
- or
- b) possibly a mathematical model for the behaviour of the phenomenon, depending on the parameter  $\mu$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either
- Reconstruction by interpolation  
atwise  $u(x_i, t_k, \mu)$   
moments  $\int \varphi_{i,k}(x, t) u(x, t, \mu) dx dt$
- or
- b) possibly a mathematical model  
phenomenon, depending on the behaviour of the parameter  $\mu$
- Reconstruction by simulation

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

a) possibly measures, either

Reconstruction by interpolation  
moments  $\int_{\Omega} \kappa(x, t) u(x, t, \mu) dx dt$

or

b) possibly a mathematical model  
phenomenon, depending on the behaviour of the parameter  $\mu$

Reconstruction by simulation

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either

Reconstruction by interpolation  
moments  $\int_{\Omega} \kappa(x, t) u(x, t, \mu) dx dt$

EIM or hyperreduction

or

- b) possibly a mathematical model of the phenomenon, depending on the parameter  $\mu$

Reconstruction by simulation

Reduced basis methods or PGD

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either pointwize  $u(x_i, t_k, \mu)$   
or moments  $\int \varphi_{i,k}(x, t)u(x, t, \mu)dxdt$

AND

- b) possibly a mathematical model for the behaviour of the phenomenon, depending on the parameter  $\mu$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either moments  $\int \varphi_{i,k}(x, t) u(x, t, \mu) dx dt$  orwise  $u(x_i, t_k, \mu)$   
Possibly polluted with errors and randomness
- AND
- b) possibly a mathematical model for the behaviour of the phenomenon, depending on the parameter  $\mu$

# Vague Statements

In order to determine  $u(x, t; \mu)$ : what do we have at end ?

- a) possibly measures, either moments  $\int \varphi_{i,k}(x, t) u(x, t, \mu) dx dt$  orwise  $u(x_i, t_k, \mu)$   
Possibly polluted with errors and randomness
- b) possibly a mathematical model for the behaviour of the phenomenon, depending on the parameter  $\mu$   
Possibly inaccurate and suffering from bias

AND

Let us assume that we have such a sequence of optimal discrete spaces  $\{X_N\}_N$

Let us assume that we have such a sequence of  
optimal discrete spaces  $\{X_N\}_N$

and measurements

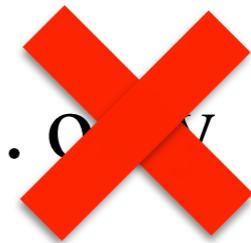
**EIM/GEIM**

# EIM/GEIM

Reconstruction from data .. only

# EIM/GEIM

Reconstruction from data .. on  $\mathcal{V}$



and a background spaces  $X_N$

# EIM/GEIM

The Empirical Interpolation Method (EIM)  
proposed in 2004 with M. Barrault, N. C. Nguyen and A. T. Patera

This approach allows to determine an “empirical” optimal set of interpolation points and/or set of interpolating functions.

In 2013, with Olga Mula, we have generalized it (GEIM) to include more general output from the functions we want to interpolate : not only pointwise values but also some moments.

recursive (greedy) definition of the functions and the interpolation points if  $\mathcal{I}_{n-1}$  is defined by

$$\mathcal{I}_{n-1}(u) = \sum_{i=1}^{n-1} \alpha_i \zeta_i$$

so that

$$\mathcal{I}_{n-1}(u)(x_j) = u(x_j)$$

then

$$\mu_n = \operatorname{argmax}_\mu \|u(\mu) - \mathcal{I}_{n-1}(u(\mu))\|$$

and

$$x_n = \operatorname{argmax}_x |u(x; \mu_n) - \mathcal{I}_{n-1}(u(\mu_n))(x)|$$

The algorithm tells you what points to choose in order to interpolate with functions in  $\mathcal{S}$

The algorithm tells you what points to choose in order to interpolate with functions in  $\mathcal{S}$

greedy !

The algorithm tells you what points to choose in order to interpolate with functions in  $\mathcal{S}$

greedy !

kind of ad hoc approach

Problem ... does not respect the positivity of the functions

idea : modify, on the fly, the basis functions

Kathrin Glau, Ladya Khoun



---

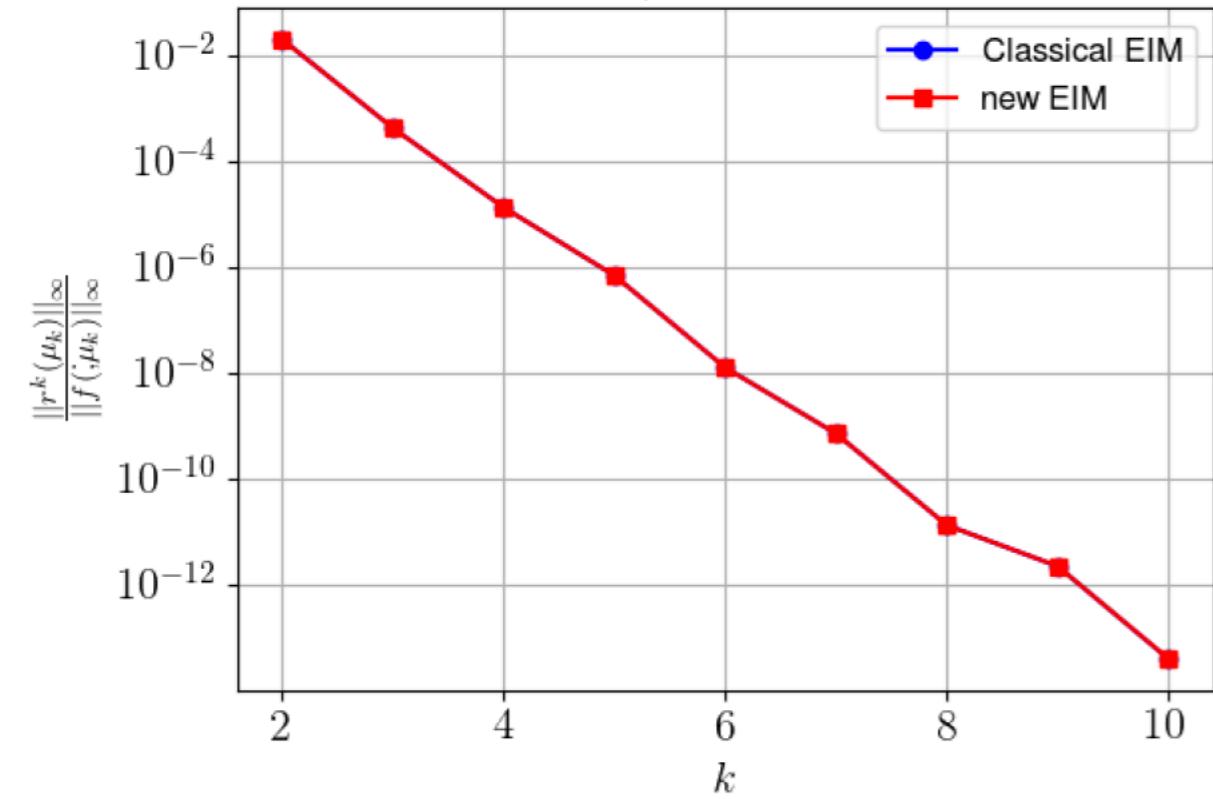
**Algorithm 3** : An iterative algorithm for solving the constrained minimization problem (3)**Input** : Stopping criteria  $\epsilon$ , maximum number of iteration  $N_{max}$ **Output** : A solution  $\{\lambda_{ij}^{k+1}, j \neq i\}$  of the problem (3)

- 1:  $n = 1$
- 2:  $(\lambda_{ij}^{k+1})^n = 0, \forall j = 1, \dots, k + 1, j \neq i$
- 3: **while** ( $n \leq N_{max}$ ) **do**
- 4:   **for** ( $j = 1, \dots, k + 1, j \neq i$ ) **do**
- 5:     Define  $(\lambda_{ij}^{k+1})^*$  as the largest real value  $\lambda_{ij}^{k+1}$  such that

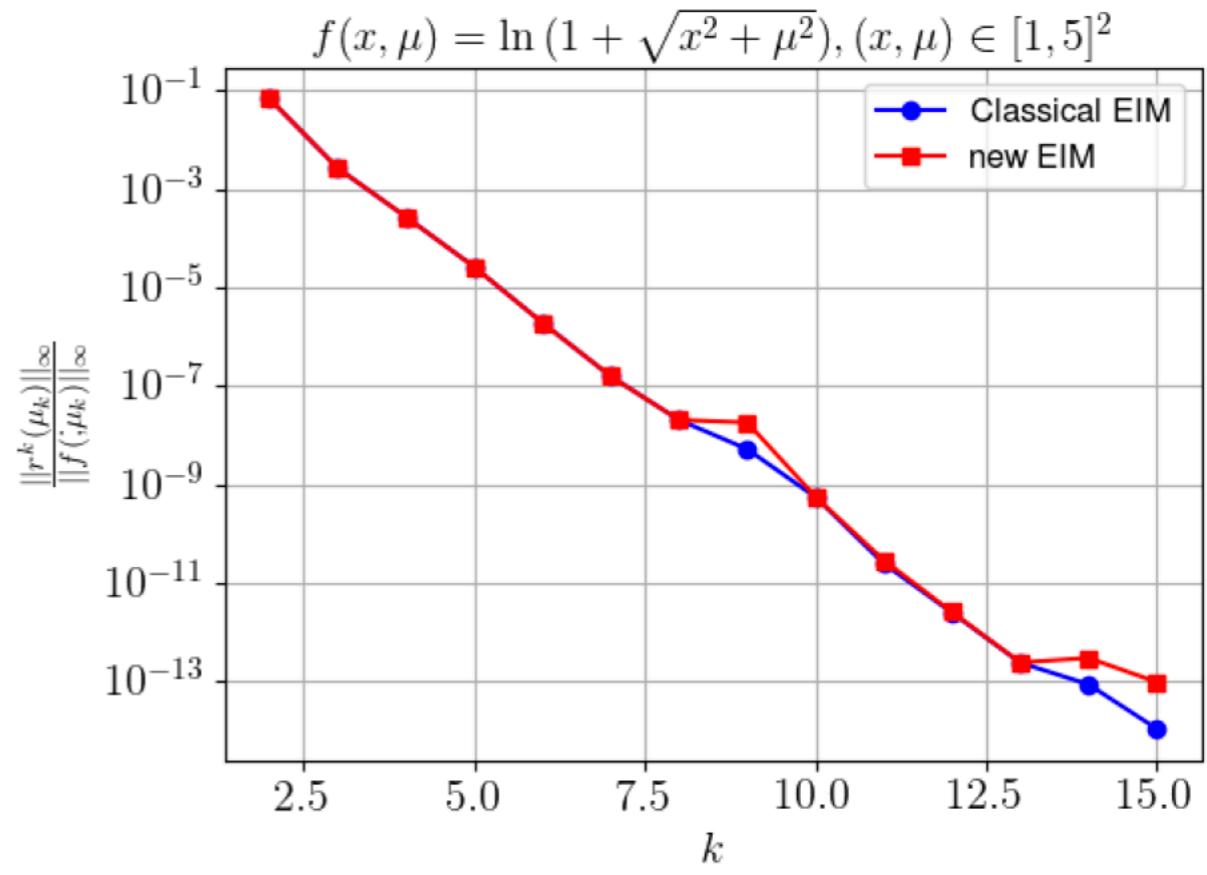
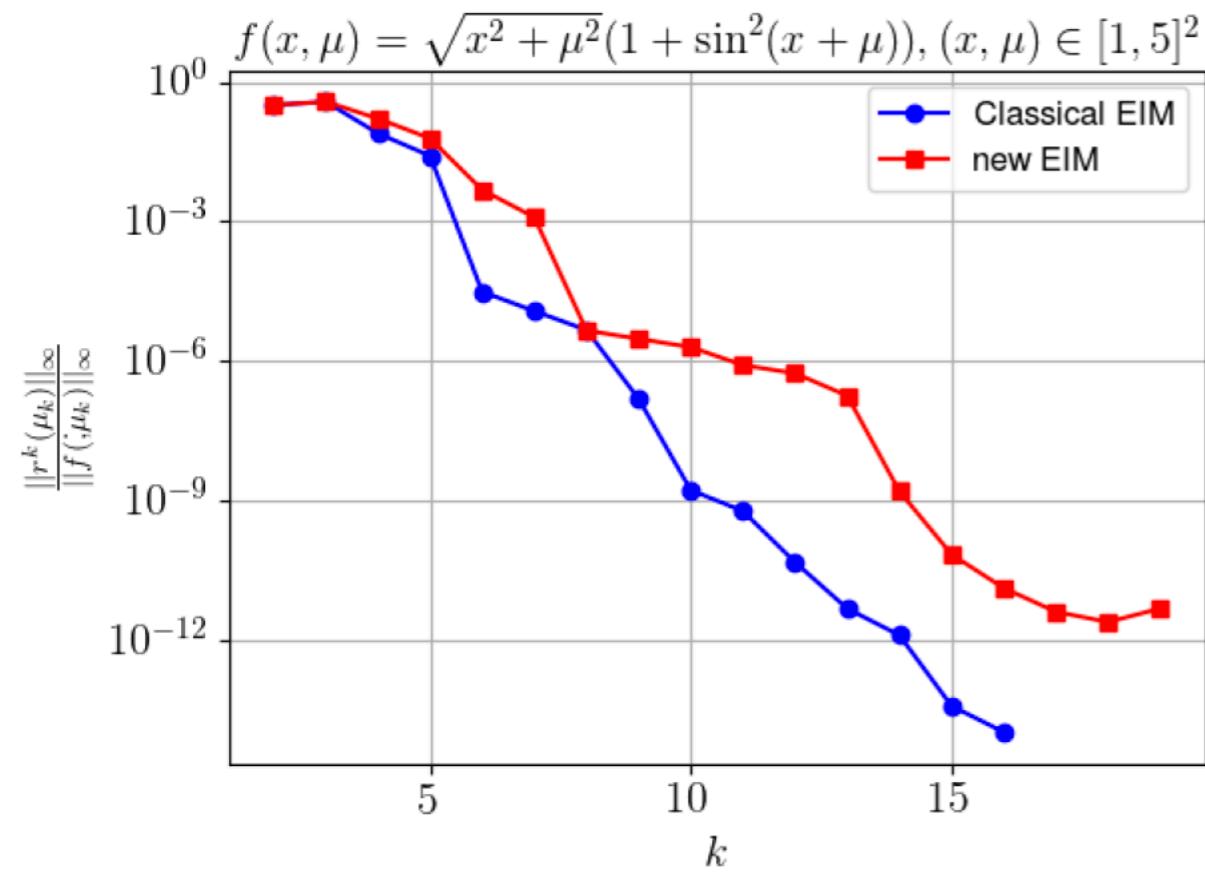
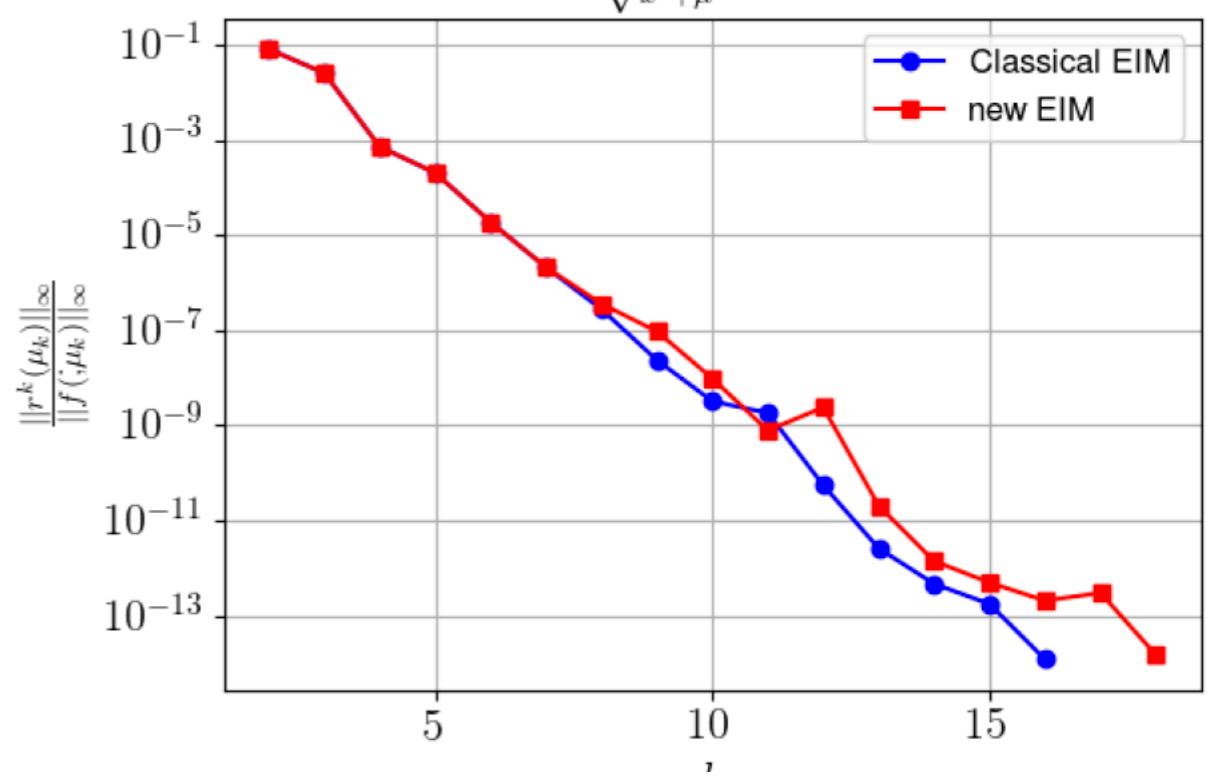
$$\gamma_i - \sum_{l=1, l \neq i, l \neq j}^{k+1} (\lambda_{il}^{k+1})^n \gamma_l - \lambda_{ij}^{k+1} \gamma_j \geq 0, \forall x \in \Omega$$

- 6:     Set  $(\lambda_{ij}^{k+1})^{n+1} = (1 - 2^{-n}) \lambda_{ij}^*$
  - 7:   **end for**
  - 8:   **if**  $(|(\lambda_{ij}^{k+1})^{n+1} - (\lambda_{ij}^{k+1})^n| \leq \epsilon |(\lambda_{ij}^{k+1})^n|, \forall j = 1, \dots, k + 1, j \neq i)$  **then**
  - 9:     **break**;
  - 10:   **end if**
  - 11:    $n = n + 1$
  - 12: **end while**
-

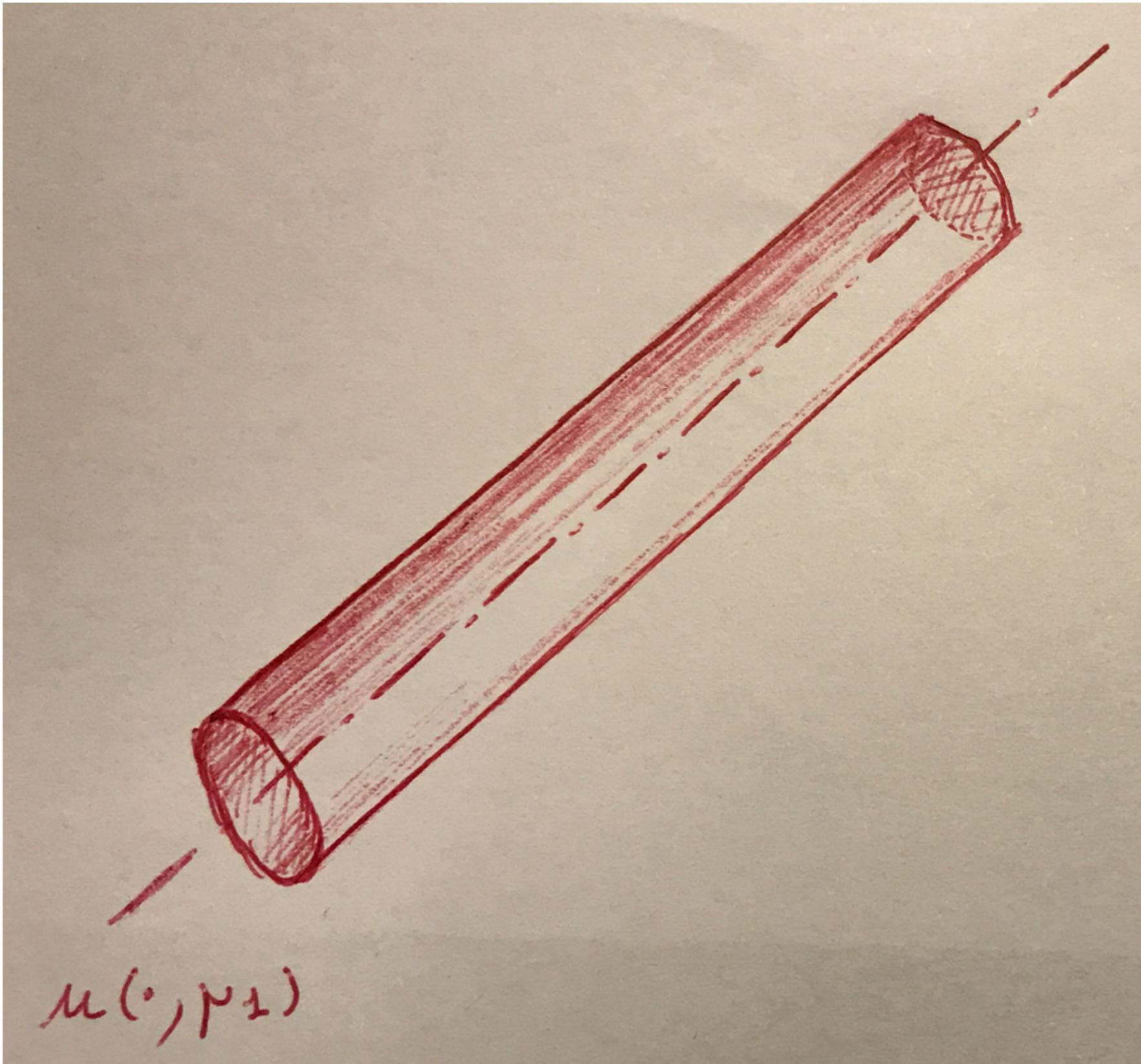
$$f(x, \mu) = \frac{1}{\sqrt{x+\mu}}, (x, \mu) \in [1, 5]^2$$



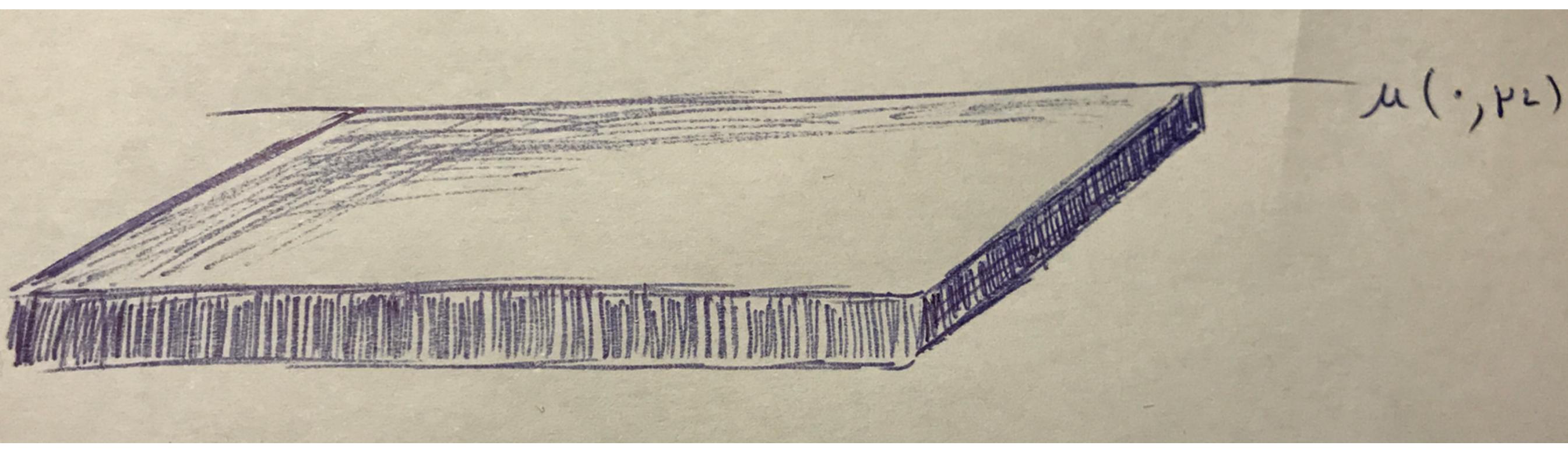
$$f(x, \mu) = \frac{1}{\sqrt{x^2+\mu^2}}, (x, \mu) \in [1, 5]^2$$



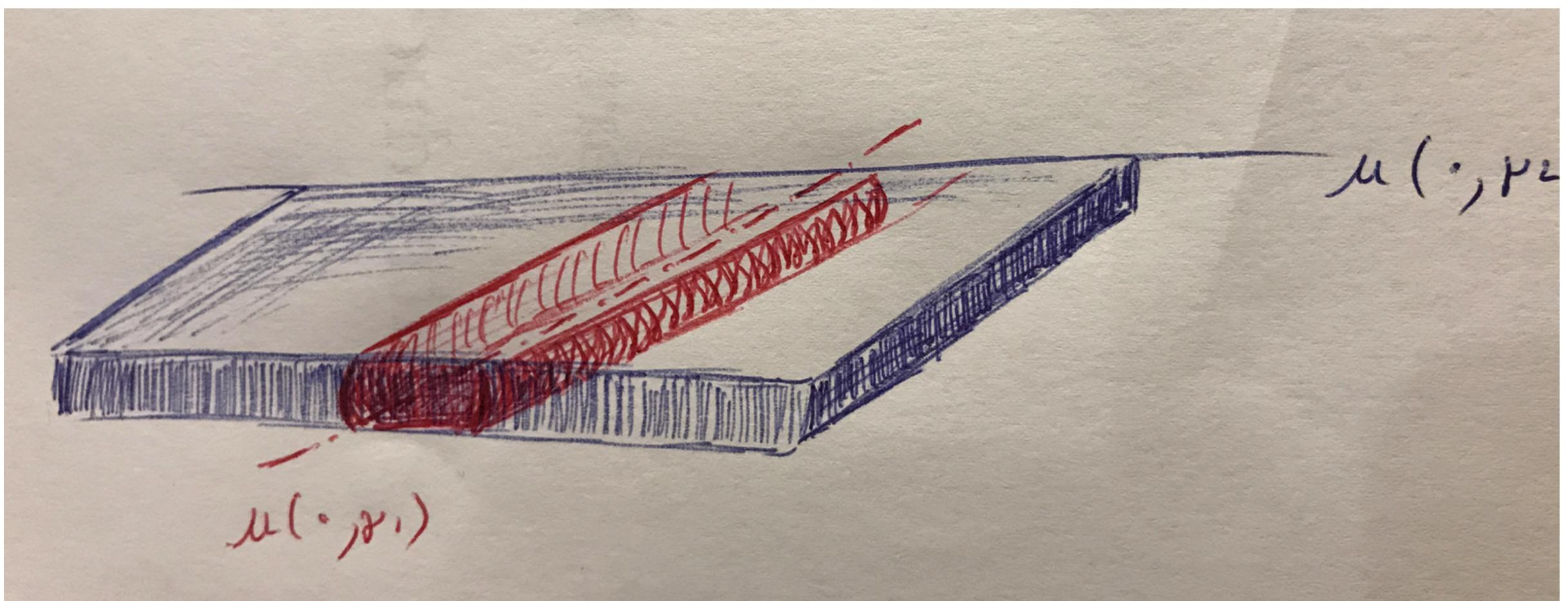
Better use of the whole family of  
optimal discrete spaces  $\{X_N\}_N$



This is the part of  $X_I$  of interest



This is the part of  $X_2$  of interest



And this is actually where we should be looking at  
 $X_1 \cap X_2$

How can we do this ?



Remember the recursive formula

Remember the recursive formula

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)](x_M)} \left[ [u(., \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)]] \right]$$

Remember the recursive formula

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)](x_M)} \left[ [u(., \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)]] \right]$$

That we better rewrite as

Remember the recursive formula

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)} \left[ [u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]] \right]$$

That we better rewrite as

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M) \right] \frac{[u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)}$$

Remember the recursive formula

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)} \left[ [u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]] \right]$$

That we better rewrite as

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M) \right] \frac{[u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)}$$

and let us introduce

Remember the recursive formula

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)](x_M)} \left[ [u(., \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)]] \right]$$

That we better rewrite as

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M) \right] \frac{[u(., \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)](x_M)}$$

and let us introduce

$$q_M = \frac{[u(., \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(., \mu_M)](x_M)}$$

Remember the recursive formula

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \frac{u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M)}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)} \left[ [u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]] \right]$$

That we better rewrite as

$$\mathcal{I}_M[u(\cdot, \mu)] = \mathcal{I}_{M-1}[u(\cdot, \mu)] + \left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(\cdot, \mu)](x_M) \right] \frac{[u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)}$$

and let us introduce

$$q_M = \frac{[u(\cdot, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)]]}{u(x_M, \mu_M) - \mathcal{I}_{M-1}[u(\cdot, \mu_M)](x_M)}$$

and remark  $q_M$  is order 1, so that

$$\mathcal{I}_M[u(.,\mu)] = \mathcal{I}_{M-1}[u(.,\mu)] + \left[ u(x_M,\mu) - \mathcal{I}_{M-1}[u(.,\mu)](x_M) \right] q_M(.)$$

$$\mathcal{I}_M[u(.,\mu)]=\mathcal{I}_{M-1}[u(.,\mu)]+\Big[u(x_M,\mu)-\mathcal{I}_{M-1}[u(.,\mu)](x_M)\Big]q_M(.)$$



$$\mathcal{I}_M[u(.,\mu)]=\mathcal{I}_{M-1}[u(.,\mu)]+\Big[u(x_M,\mu)-\mathcal{I}_{M-1}[u(.,\mu)](x_M)\Big]q_M(.)$$

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \underbrace{\left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M) \right]}_{\text{This quantity is small}} q_M(.)$$

This quantity is small

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \underbrace{\left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M) \right]}_{\text{This quantity is small}} q_M(.)$$

hence, we can write

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \alpha_M q_M(.)$$

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \underbrace{\left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M) \right]}_{\text{This quantity is small}} q_M(.)$$

This quantity is small

hence, we can write

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \alpha_M q_M(.)$$

or again

$$\mathcal{I}_M[u(., \mu)] = \sum_n \alpha_n q_n(.)$$

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \underbrace{\left[ u(x_M, \mu) - \mathcal{I}_{M-1}[u(., \mu)](x_M) \right]}_{\text{This quantity is small}} q_M(.)$$

This quantity is small

hence, we can write

$$\mathcal{I}_M[u(., \mu)] = \mathcal{I}_{M-1}[u(., \mu)] + \alpha_M q_M(.)$$

or again

$$\mathcal{I}_M[u(., \mu)] = \sum_n \alpha_n q_n(.)$$

where the  $\alpha_n$  are going to zero as  $n \rightarrow \infty$

with every  $q_n$  of order 1.

So we want to use this information that  
the  $\alpha_n$  are going to zero as  $n \rightarrow \infty$

**This gives rise to the Constrained Stabilized (G)EIM**

from J.P. Argaud, B. Bouriquet, H. Gong, Y. Maday, O. Mula (\*)

(\*) in *Stabilization of (G)EIM in presence of measurement noise: application to nuclear reactor physics*

# Constrained Stabilized EIM

We write

$$u_N = \sum_n \alpha_n q_n$$

so as to solve

$$\min_{\alpha_n} \sum_i |u_N(x_i) - u(x_i)|^2$$

under the constraint that

$$|\alpha_n| \leq \varepsilon_n$$

# Constrained Stabilized GEIM

We write

$$u_N = \sum_n \alpha_n q_n$$

so as to solve

$$\min_{\alpha_n} \sum_i |\sigma_i(u_N) - \sigma_i(u)|^2$$

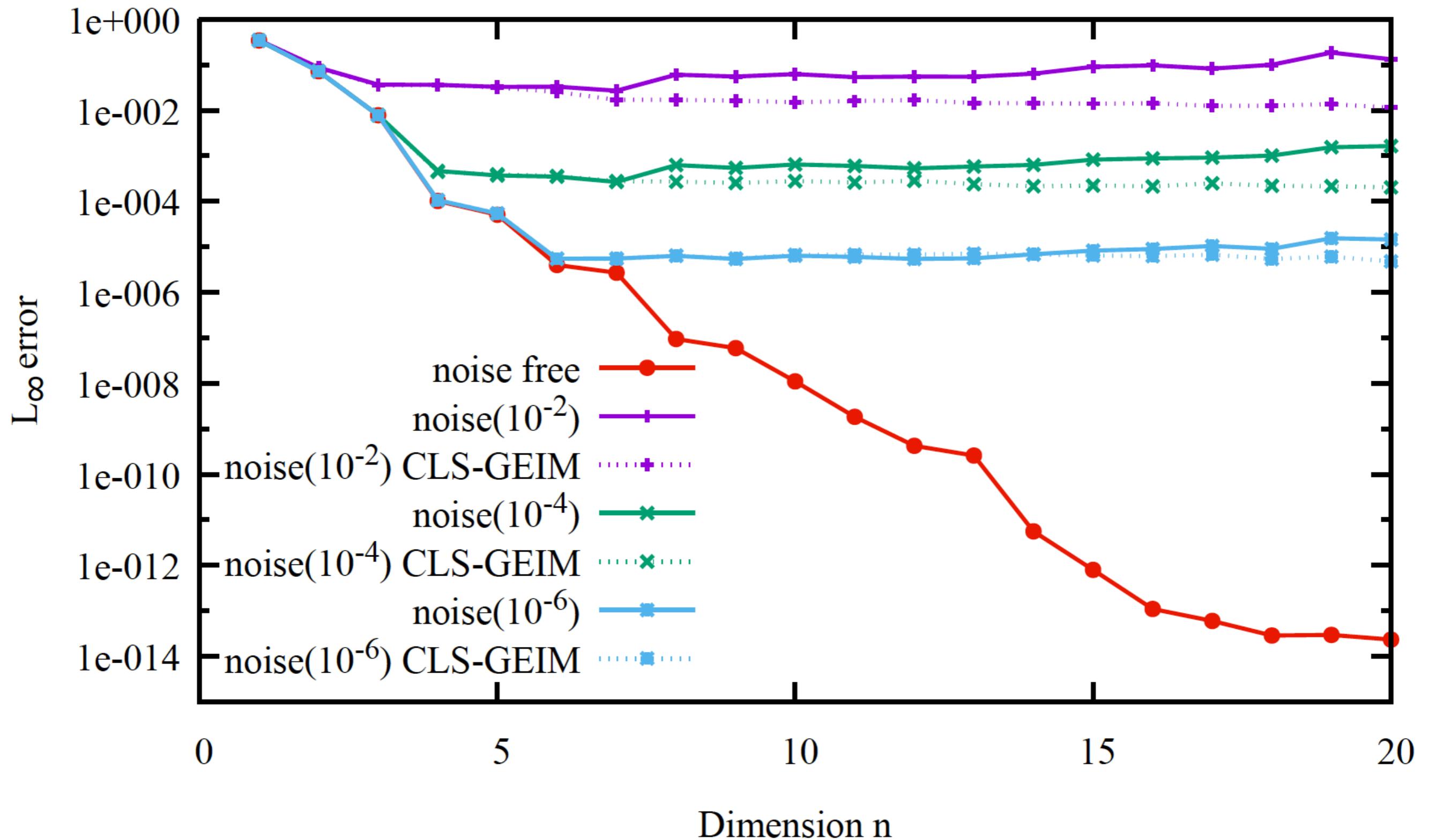
under the constraint that

$$|\alpha_n| \leq \varepsilon_n$$

The main interest is with noisy data

Assume that  $u(x_i)$  (or the  $\sigma_i(u)$ ) are polluted with some (random) noise  $\eta_i$

The CS approximation allows to minimize the effect of the noise



The data are polluted with noise

$$\forall i = 1, \dots, n, \sigma_i(\mathcal{J}_n[u]) = \sigma_i(u) + \varepsilon_i$$

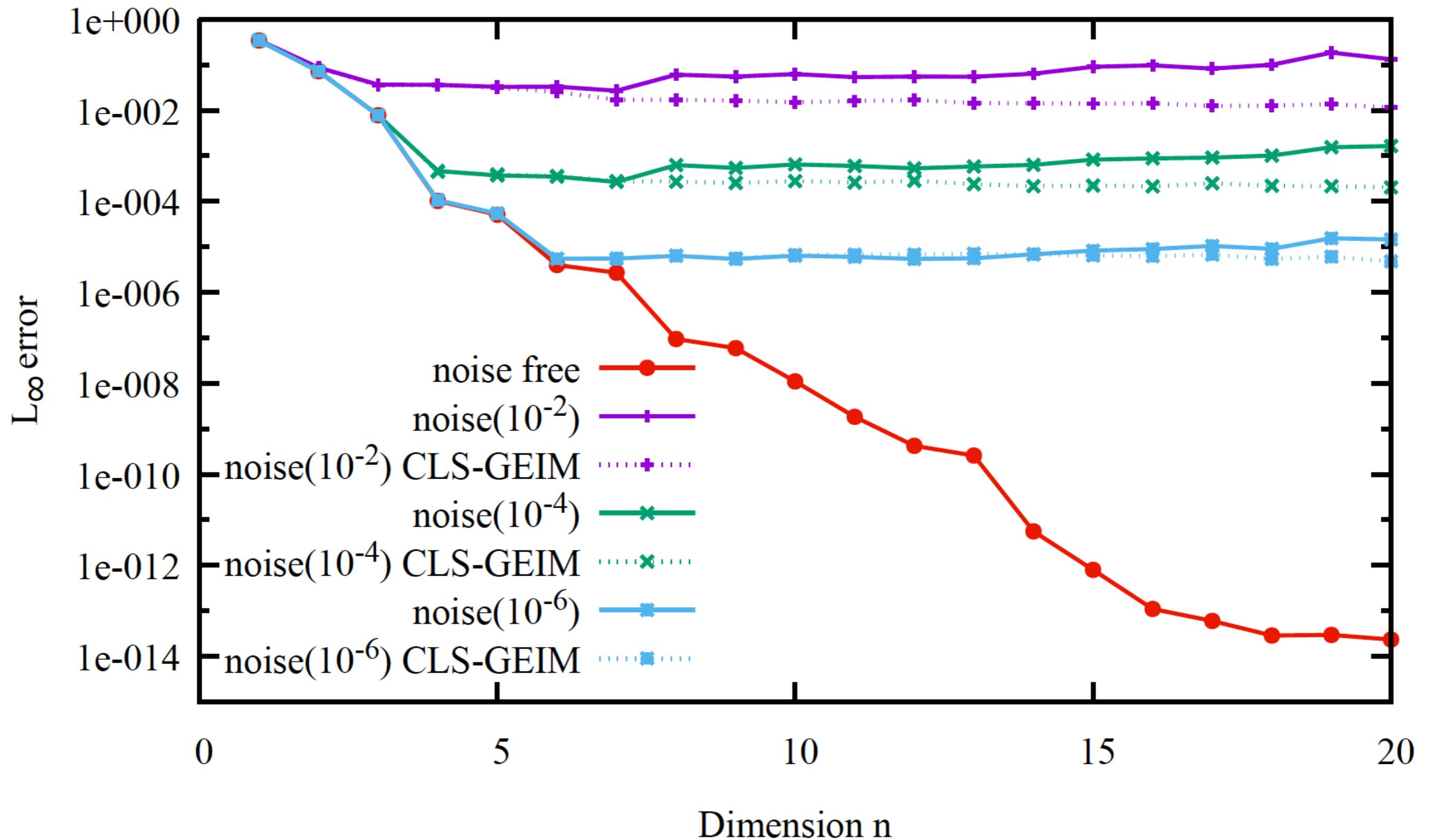
this leads to a polluted reconstruction

$$\mathcal{J}_n[u, \varepsilon] = \sum_{j=1}^n \tilde{\beta}_j \varphi_j, \text{ such that } \forall i = 1, \dots, n, \sigma_i(\mathcal{J}_n[u, \varepsilon]) = \sigma_i(u) + \varepsilon_i$$

And of course now, the error, scales like

$$\|u - \mathcal{J}_n[u, \varepsilon]\|_{\mathcal{X}} \leq (1 + \Lambda_n) \inf_{v_n \in X_n} \|u - v_n\|_{\mathcal{X}} + \Lambda_N \max_{i=1, \dots, n} |\varepsilon_i|$$

This is what we see here



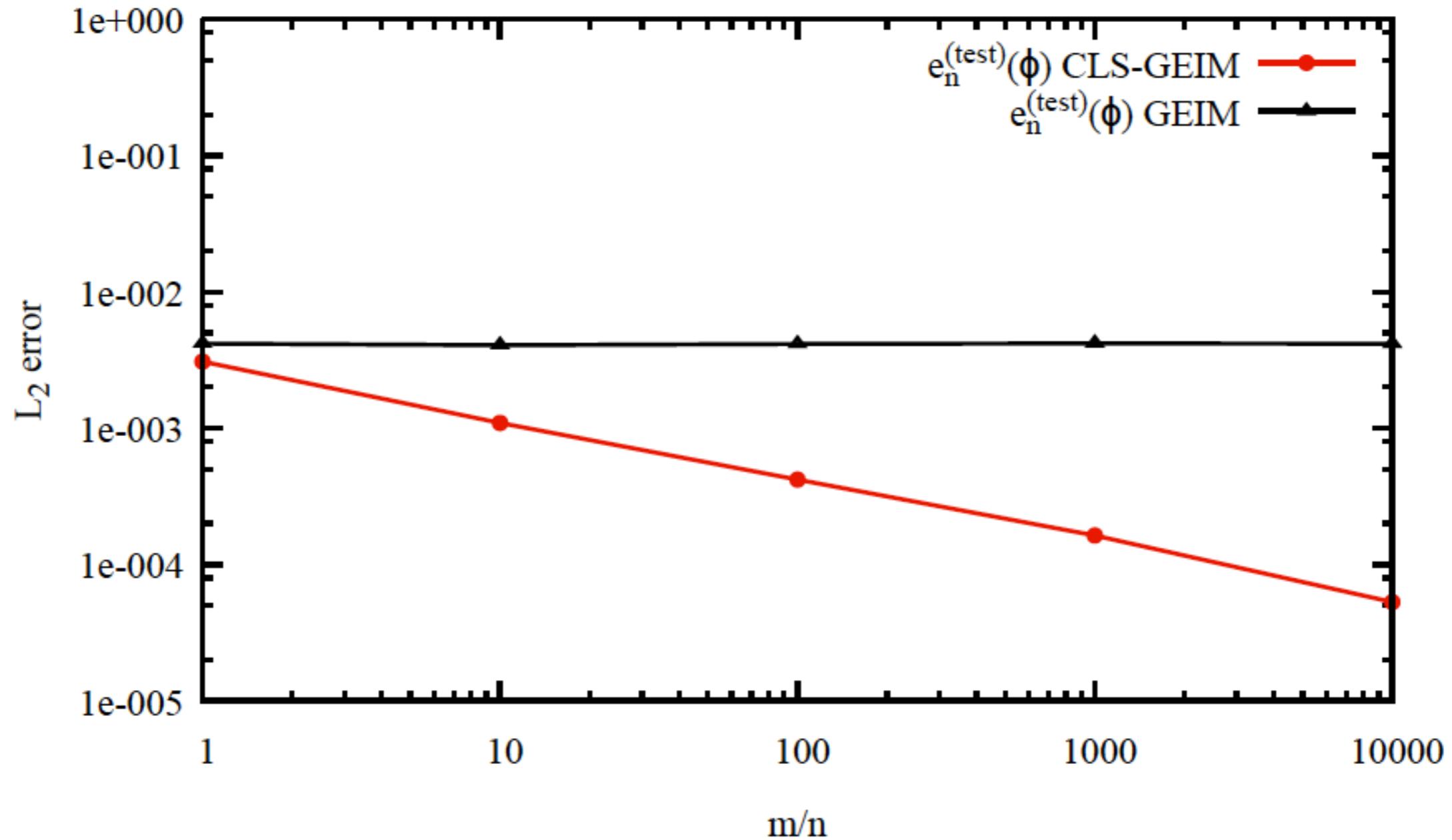


Fig. 12: CS-GEIM with different  $n/m$  ratio, the input noise level is  $10^{-2}$ , with the function  $g(x, \mu) \equiv ((x_1 - \mu_1)^2 + (x_2 - \mu_2)^2)^{-1/2}$ , the error converges with  $\sim n^{-\frac{1}{2}}$ .

Let us assume that we have such a family of optimal discrete spaces  $\{X_N\}_N$

and the model

Reduced basis method : approximation of a PDE

With such a  $X_N$ ...

Perform a Galerkin approximation

With domain decomposition : Reduced basis element method

Much to say : off-line, on-line

Requires in the offline process invasive immersion in the code

Requires in the offline process invasive immersion in the code

NIRB

Requires in the offline process invasive immersion in the code

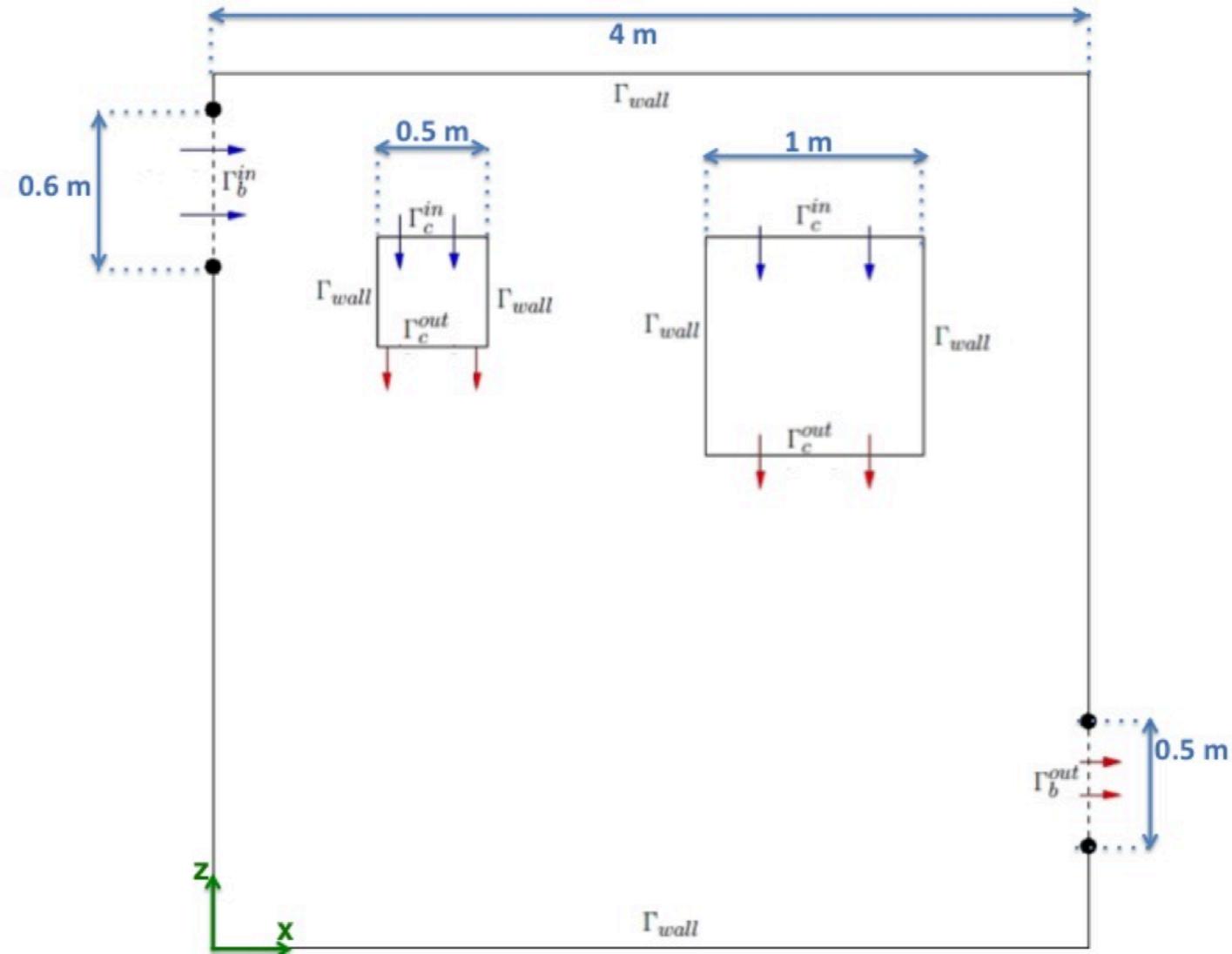
NIRB

Non Invasive Reduced Basis Method

with Rachida Chakir

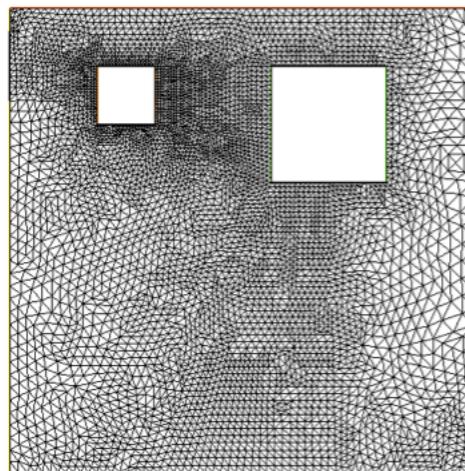


# Cooling system of electronic devices

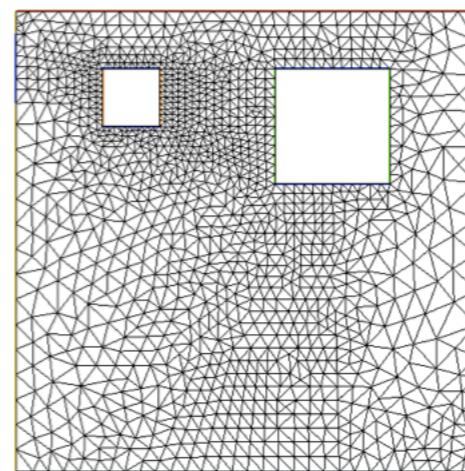


The varying parameters are the velocities  $v_b \in [0.5; 2]$ ,  $v_c \in [0.1; 0.4]$  (in mm/s), the imposed temperatures  $\theta_b \in [288; 292]$  and  $\theta_c \in [295; 315]$  (in Kelvin). For simplicity we will also denote by  $\sigma = (v_b, \theta_b, v_c, \theta_c)$  the set of parameters.

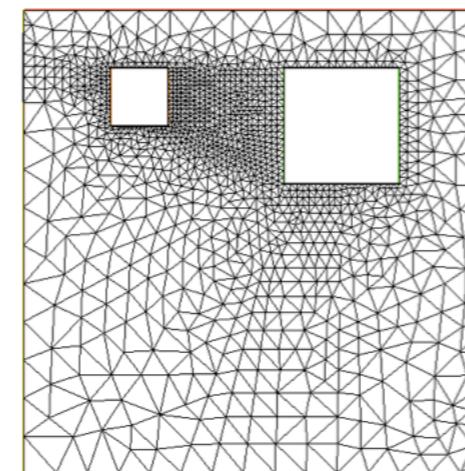
# Cooling system of electronic devices



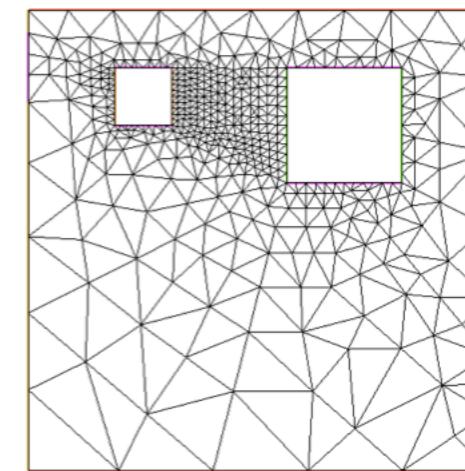
$\mathcal{T}_h$   
 $\mathbb{P}2$  Ndof = 20213



$\mathcal{T}_{H_1}$   
 $\mathbb{P}2$  Ndof = 5148

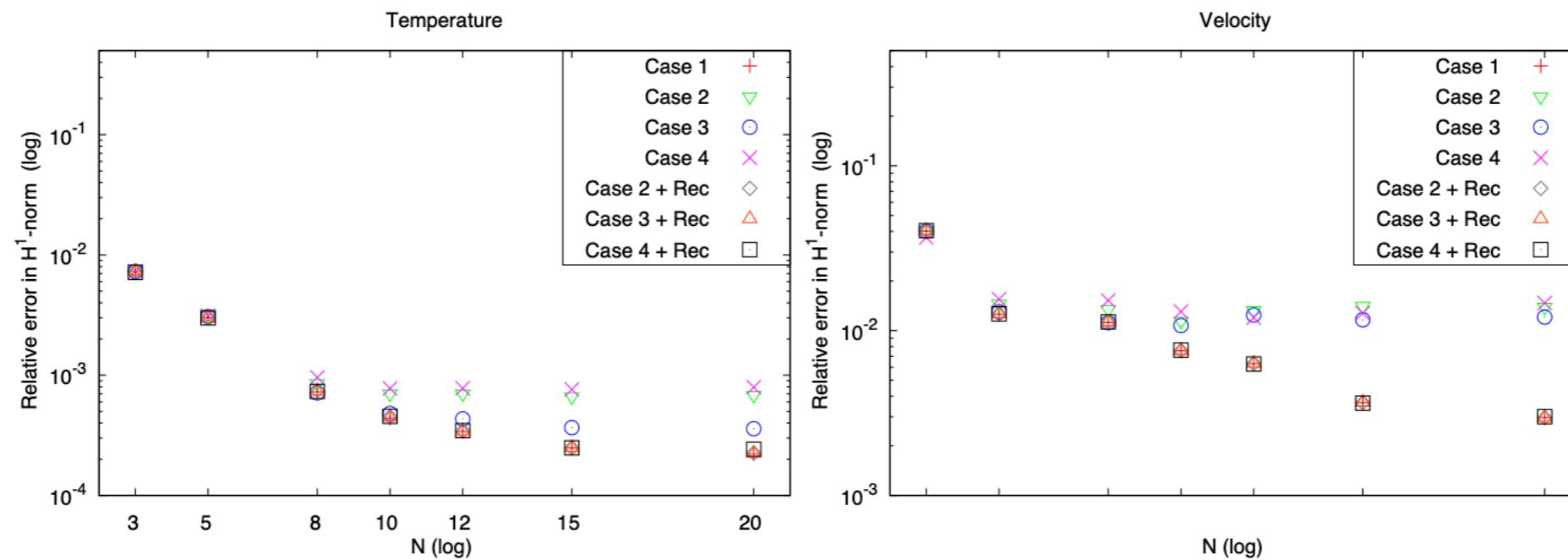
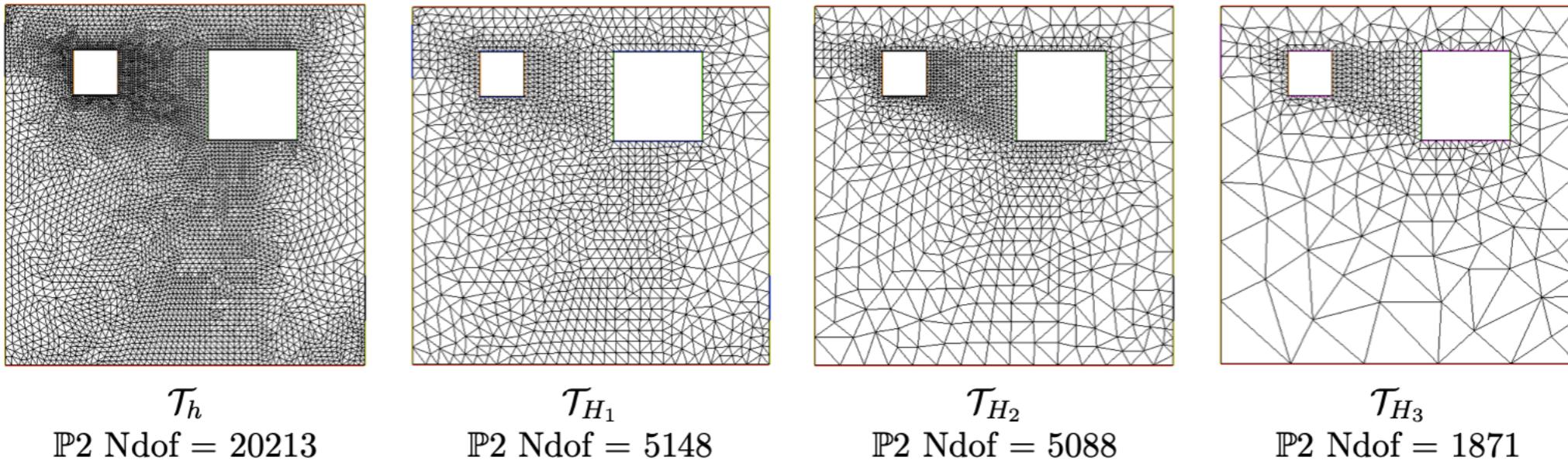


$\mathcal{T}_{H_2}$   
 $\mathbb{P}2$  Ndof = 5088



$\mathcal{T}_{H_3}$   
 $\mathbb{P}2$  Ndof = 1871

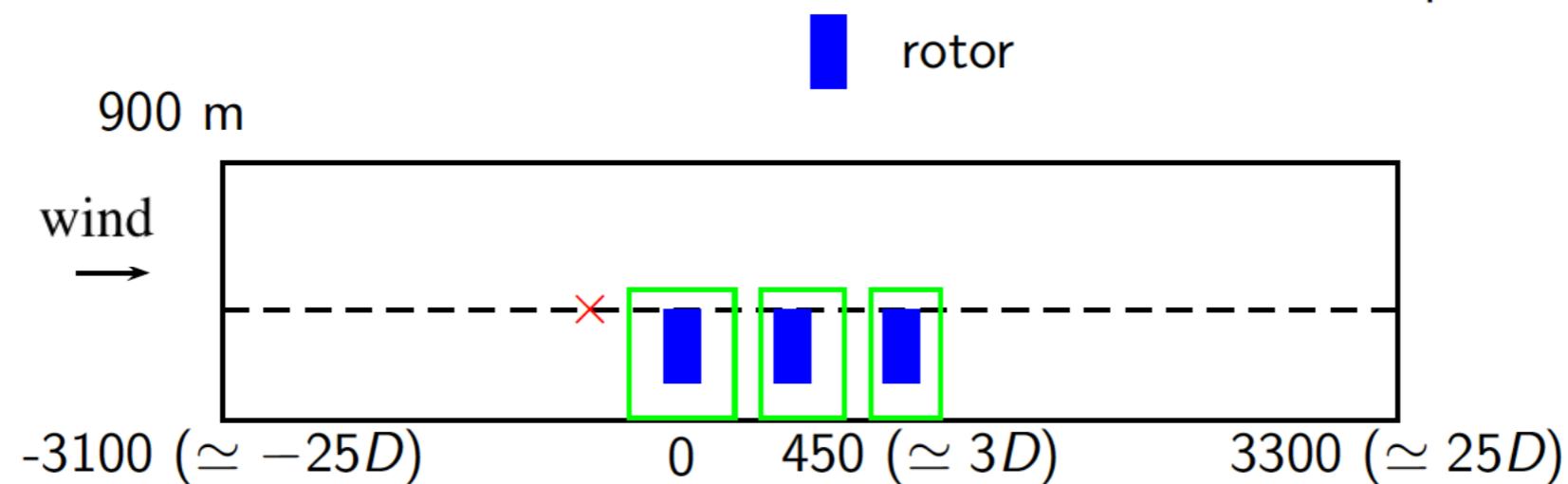
# Cooling system of electronic devices



Rectification replaced by  
constraints on the coefficients

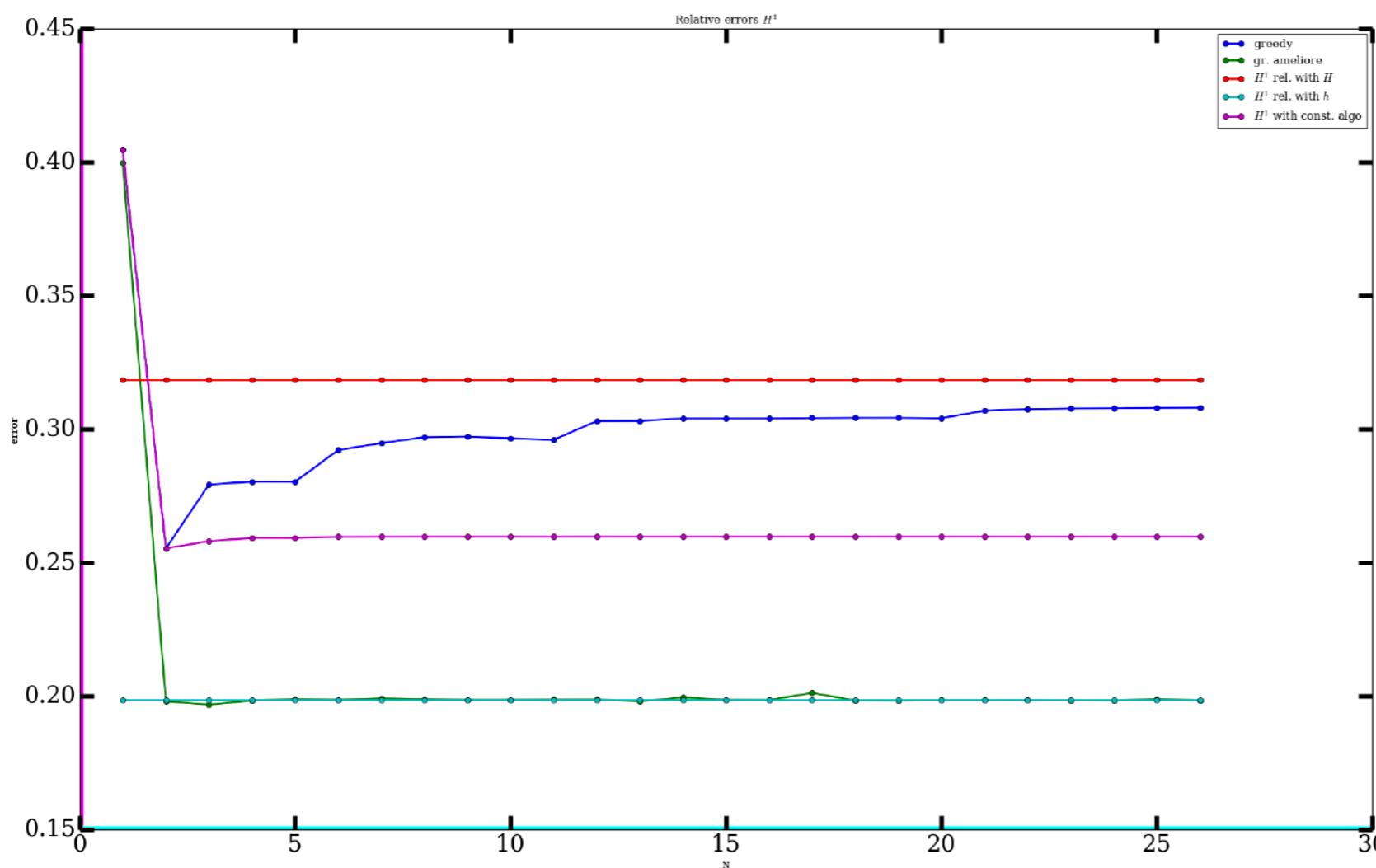
with Elise Grosjean

Rectification replaced by  
constraints on the coefficients



with Elise Grosjean

# Rectification replaced by constraints on the coefficients



with Elise Grosjean

## Conclusion

Full use the structure of the Kolmogorov n-width

- preserve the positivity
- constraints on the coefficients
- NIRB

Thanks

Questions/remarks ??